# Reliability of Complex Hierarchical Fault Tolerance Systems

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## 1 Introduction and Motivation

Up to day complex technical systems are characterized by the following main properties: hierarchical structure and build-in system of the state control.

Hierarchical structure means that the system consists of subsystems which also are divided on sub-subsystems etc. up to the lowest non-divisible components, to which we will refer as to *units*. Usually, hierarchical structures have the property that the primary failures arise mainly at the lowest (elementary) level of the system. Then gradually developing these failures it leads to the failure of blocks and subsystems of higher levels containing these failed elements. This means that the failures of the whole system are mainly not instantaneous, but also gradual, i.e. from an absolutely good state to complete failure state the system goes through several intermediate states (fault stages). Such failures may change the state of the system and the quality of its operation, but do not necessarily lead to complete system failure. From an abstract point of view these systems could be described as multi-state reliability systems. A present-day state of the subject can be found in (1). Some models of multi-state hierarchical systems are considered in (2), (3).

The presence of build-in system of control (SoC) leads to the fact that such systems are called Fault Tolerance Systems (FTS). The SoC detects the faults and corrects them itself or gives a signal about the necessity of repair. Some times and resources (costs, or reward loses) are incurred for the repair. In case that a system (subsystem or any block) is put under repair, any further degradation is assumed impossible and after a repair it is assumed to be "as good as new". This means that the reliability of the system is partially controllable. Various repair policies are possible after a gradual failure in some part of a system is detected, e.g. to replace the whole system, only the failed element, or repair just some structural parts of the system. The case of the whole system repair is considered in (2), and the case of the failed unit repair only is considered in (3). Different kind of preventive maintenance are considered by many authors (see Gertsbakh (4) and the references therein).

Two main characteristics are common in the reliability studies: the life-time of the system, and its steady state characteristics under some assumptions about repair process. The ways to evaluate these characteristics depend on the approach to the following two aspects: probabilistic and structural. The probabilistic aspect deals with calculation of the system states probabilities, and uses them in reliability calculations. The structural aspect considers kind of aggregated direct evaluation of reliability characteristics for any given structure of a particular system.

The structure of system itself and its failure set are individual for each system (or class of systems) and should be considered for each system individually. Here we will not consider special structural properties of systems and focus on statistical properties of complex hierarchical systems reliability.

In this paper we propose a general approach to describe, model and evaluate the most common reliability characteristics of complex hierarchical systems with various types of gradual failures and different repair policies.

## 2 A Mathematical Model

Consider some *complex hierarchical multi-component system* subject to *gradual failures* of different types as described above. To specify the states space of the system and to define appropriate process describing its behavior let us introduce vector index $k = (i_1, i_2, \ldots, i_{L(k)})$ which determine each unit of the system as belonging to appropriate chain of blocks at any level with level of $k$-th unit denoted by $L(k)$. Denote also by $\mathcal{K}$ the set of these indices (and appropriate units) with $K = \#(\mathcal{K})$ number of units. Then the *states space* of the system can be represented as $E = \{\mathbf{x} = (x_k : \ k \in \mathcal{K})\}$, where for any $k \in \mathcal{K}$ the integer $x_k$ represents the state of the $k$-th unit in sense of its reliability. It can take different values, depending on its type, $x_k \in \{0, 1, \dots m_k\}$, where the exhausted state of $k$-th unit is denoted by $m_k$.

We model the system functioning with a finite state multi-dimensional Markov process

$$\mathbf{X} = \{X_k(t) : \ k \in \mathcal{K}, \ t \geq 0\},$$

with set of states $E$, which should be specified for any particular system. Denote also by $N$ the set of all normal system states, by $D$ the set of dangerous states, and by $F$ the set of the full system breakdown states. Moreover it is supposed that these subsets contain "boundary" sub-subsets $\Gamma_{ND}$, $\Gamma_{NF}$ and $\Gamma_{DF}$, such that the transition to the states in $D$ and $F$ is possible only from the states of these subsets.

Additional assumption concerns the structure of transition intensities of such a process. The specific of the reliability models make it reasonable to suppose that in the case that gradual failure arises the process can jump only in neighboring states, and some function $f = f_k(\mathbf{x})$ determines the states to which the process goes in the case of a fault in $k$-th unit is detected and appropriate repair is completed. The repair function $f$ can be given by different ways. The cases in which as repair resulted in the whole system being renovated and only an unit renovated were considered in (2), (3). Another repair policies are also possible, for example, some subsystem of given level could be renovated as a repair result. This means that the transition intensities have the following form

$$a(\mathbf{x}, \mathbf{y}) = \begin{cases} \alpha_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \ \mathbf{x}, \mathbf{y} \in N, \\ \lambda_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \ \mathbf{x} \in \Gamma_N, \ \mathbf{y} \in D, \\ \beta_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \ \mathbf{x}, \mathbf{y} \in D, \\ \mu_k(\mathbf{x}) & \text{for } \mathbf{y} = f(\mathbf{x}), \ \mathbf{x} \in D, \\ \gamma_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \ \mathbf{x} \in \Gamma_N, \ \mathbf{y} \in F, \\ \nu_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \ \mathbf{x} \in \Gamma_N, \ \mathbf{y} \in F. \end{cases} \tag{1}$$

Here and later notation $\mathbf{e}_k$ means a unit vector with 1 at $k$-th position and zeros elsewhere. Different constrains to the state space $E$ and/or the failure set $F$ of this process and various assumption about dependence of transition intensities on the state give an opportunity to model a number of particular cases.

The Kolmogorov's system of differential equations for the time dependent probabilities (TDP) and appropriate algebraic system of equation for steady state probabilities (SSP) are given. In general these systems of equations give the possibility to investigate (at least numerically) both the stationary and the time dependent reliability characteristics of the system. In this paper we consider two cases, which admit close form and algorithmic solution, the case of Whole System Repair Policy (SRP) and the case of only a Unit Repair Policy (URP) after he failure detection. To consider these special cases we define a partial order in $E$ as follows:

$$\mathbf{x} < \mathbf{y}, \qquad \text{if } x_k \leq y_k \quad \text{for all } k \in \mathcal{K} \quad \text{and at least for one } x_k < y_k,$$

and we will use the following notations.

$$\begin{aligned}
\mathbf{x}_k(i) & \quad \text{is the vector } \mathbf{x} \text{ with } k\text{-th component equals to } i \text{, i.e. } \mathbf{x}_k(i) = \mathbf{x} + (i - x_k)\mathbf{e}_k; \\
\Gamma_r & \quad \text{is a hyper-plane with any of } r\text{-th components does not equal zero;} \\
p_r(\mathbf{x}) & \quad \text{is a monotone path from state } \mathbf{0} \text{ to state } \mathbf{x} \text{ through hyper-plane } \Gamma_r \\
\alpha_i & = \alpha(\mathbf{x}_{i-1}, \mathbf{x}_i) \text{is the label of unit, which failure leads to transition from state } \mathbf{x}_{i-1} \text{ to state } \mathbf{x}_i; \\
g(p(\mathbf{x})) & = \prod_{1 \leq i \leq l} \frac{\lambda_\alpha(\mathbf{x}_{i-1})}{\gamma(\mathbf{x}_i)}; \\
P_r(\mathbf{x}) & \quad \text{is the set of all monotone paths from state } \mathbf{0} \text{ to the state } \mathbf{x} \text{ ;} \\
G_r(\mathbf{x}) & = \sum_{p \in P_r(\mathbf{x})} g(p(\mathbf{x})) = \sum_{k:\ x_k > 0} \frac{\lambda_k(\mathbf{x} - \mathbf{e}_k)}{\gamma(\mathbf{x})} G_r(\mathbf{x} - \mathbf{e}_k) \quad \text{with} \quad G(\mathbf{0}) = 1.
\end{aligned} \tag{2}$$

## 3 FTS under Whole System Repair Policy

To simplify the model we will not make a distinction between the transition intensities inside the sets $N,\ D,\ F$ and between these sets themselves. We use for these the notation $\lambda_k(\mathbf{x})$ and $\mu_k(\mathbf{x})$, and denote by $\gamma(\mathbf{x})$ the total output intensity from the state $\mathbf{x}$. For the system under whole SRP the repair function is $f_k(\mathbf{x}) = \mathbf{0}$ and therefore accordingly to the remark above the transition intensities take the form

$$a(\mathbf{x}, \mathbf{y}) = \begin{cases} \lambda_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \\ \mu(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{0}. \end{cases} \tag{3}$$

Both the system of differential equations for the TDP and the system of algebraic equations for SSP of the process with transition intensities (3) are given. The solution these systems can be represented in closed form. The SSP is represented by the following theorem

**Theorem 1.** *The SSP of the FTS under SRP after failure detection have the form*

$$\pi_r(\mathbf{x}) = \left[\sum_{\mathbf{x}\in E} G(\mathbf{x})\right]^{-1} G_r(\mathbf{x}), \quad \textit{for any} \quad \mathbf{x} \in \Gamma_r \subset E. \qquad \Box \tag{4}$$

The reliability function of the system coincides with the tail of the distribution for the time to first entrance of the process $X(t)$ into the failure set $F$. This distribution can be found from Kolmogorov's differential equations for TDP with initial condition $\pi(\mathbf{0}; 0) = 1$, where all failure states are absorbing states, $\lambda(\mathbf{x}) = \mu(\mathbf{x}) = 0$ for all $\mathbf{x} \in F$. In terms of Laplace transforms of TDP $\tilde{\pi}(\mathbf{x}; s)$ this system is turned into the system of algebraic equations, whose solution is used in the following theorem, where we denote $\tilde{\gamma}(\mathbf{x}, s) = s + \gamma(\mathbf{x})$, and in the appropriate way change the notation for functions $g(\mathbf{x})$ and $G(\mathbf{x})$ in (2) by $\tilde{g}(\mathbf{x}, s)$ and $\tilde{G}(\mathbf{x}, s)$.

**Theorem 2.** ((2)) *The reliability function of FTS under SRP is*

$$R(t) = 1 - \pi_F(t) = 1 - \sum_{\mathbf{x}\in F} \pi(\mathbf{x}; t), \tag{5}$$

*and* $\pi(\mathbf{x}; t)$ *has the Laplace transform*

$$\tilde{\pi}(\mathbf{x}; s) = \left[\tilde{\gamma}(\mathbf{0}; s) - \sum_{\mathbf{x}\in E\setminus\mathbf{0}} \mu(\mathbf{x})\tilde{G}(\mathbf{x}; s)\right]^{-1} \times \tilde{G}(\mathbf{x}; s), \quad \mathbf{x} \in E. \qquad \Box \tag{6}$$

## 4 FTS under Unit Repair Policy

Consider now the FTS under URP. The repair function for the case under consideration is $f_k(\mathbf{x}) = \mathbf{x}_k(0)$ and so in terms of the notations (2) the transition intensities (1) are

$$a(\mathbf{x}, \mathbf{y}) = \begin{cases} \lambda_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x} + \mathbf{e}_k, \\ \mu_k(\mathbf{x}) & \text{for } \mathbf{y} = \mathbf{x}_k(0), \end{cases} \tag{7}$$

With these transitions intensities we obtain the Kolmogorov's system of differential equations for the TDP and appropriate system of equations for SSP of the process. The SSP for this model can be given in algorithmic form and represented in the following theorem.

**Theorem 3.** *The SSP for the FTS under URP are a limit of successive approximations*

$$\pi_r(\mathbf{x}) = \lim_{n\to\infty} \pi_r^{(n)}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_r \subset E \tag{8}$$

*given by the formulas*

$$\pi_r^{(n+1)}(\mathbf{x}) = \sum_{k:\ x_k>1} \frac{\lambda_k(\mathbf{x}-\mathbf{e}_k)}{\gamma(\mathbf{x})}\pi_r^{(n)}(\mathbf{x}-\mathbf{e}_k) + \sum_{k:\ x_k=1} \frac{\lambda_k(\mathbf{x}-\mathbf{e}_k)}{\gamma(\mathbf{x})}\pi_{r-1}^{(n)}(\mathbf{x}-\mathbf{e}_k) + \\ + \sum_{k:\ x_k=0} \sum_{1\le i\le m_k} \frac{\mu_k(\mathbf{x}_k(i))}{\gamma(\mathbf{x})}\pi_{r+1}^{(n)}(\mathbf{x}_k(i)), \qquad \mathbf{x}\in\Gamma_r, \tag{9}$$

*with the initial approximation, defined by the formula (4) in the theorem 1.* $\square$

Analogously to the Theorem 2 in **Theorem 4** the reliability function of the FTS under URP in terms of Laplace transforms of TDP is presented.

Recursive formulas (2) and (9) provide algorithms for calculation of the SSP and the Laplace transforms of the TDP. In the talk we discuss algorithms for calculation of the SSP. The algorithms for calculation of the Laplace transforms of the TDP follow similar procedure.

## 5 Examples

As an example we consider the homogeneous hierarchical system with $K$ units, each of which may pass only through two stages of reliability ($m = 1$). Assume, that in this case the failure and repair intensities are $\lambda_k(\mathbf{x}) = \lambda, \quad \mu_k(\mathbf{x}) = \mu$, and introduce the parameter $\rho = \mu/\lambda$.

Due to homogeneity the process $\mathbf{X}$ admits an aggregation, and so the SSP coincide for any $\mathbf{x}$ in each hyper-plane and for the model under SRP they are

$$\pi(\mathbf{x}) = \pi_r = C \prod_{1\le i\le r} \frac{i}{K-i+i\rho} \qquad \text{for any} \quad \mathbf{x}\in\Gamma_r$$

with the normalizing constant $C$ equals

$$C = \left[\sum_{0\le r\le K} \binom{K}{r} \prod_{1\le i\le K} \frac{i}{K-i+i\rho}\right]^{-1}.$$

For the FTS under URP we aggregate the states in each hyper-plane $\Gamma_r$, and after the aggregation the process $\mathbf{X}$ is reduced to a birth and death process with the phase states $\{0, 1, \ldots, K\}$. The SSP's $\hat{\pi}_r = \mathbf{P}\{\mathbf{X}\in\Gamma_r\}$ of this process can be find in closed form, and they are:

$$\hat{\pi}_r = (1+\rho)^{-1}\rho^{K-r}.$$

The system failure probabilities for the cases if only the failure of all units leads to the system failure, and if failure of any unit causes the system failure are given for both models. The numerical results for both model with $K = 4$ are presented in the paper.

## References


[1] A. Lisniansky, G. Levitin (2003) *Multi-State System Reliability. Assessment, Optimization and Application*. World Scientific. New Jersey, London, Singapore, Hong Kong, 358p.

[2] Dimitrov B.,Rykov V., Stanchev P. (2002) On Multi-State Reliability Systems. In: *Proceedings MMR-2002*. Trondheim (Norway) June 17-21, 2002.

[3] Rykov V., Dimitrov B.(2002) On Multi-State Reliability Systems. In: *Applied Stochastic Models and Information Processes*. Proceedings of the International Seminar, Petrozavodsk, Sept. 8-13. Petrozavodsk, 2002, pp. 128-135. See also http://www.jip.ru/2002-2-2-2002.htm

[4] Gertsbakh I. (2000) *Reliability theory with application to preventive maintenance*. Springer-Verlag.